4.7.3 Flow with pressure gradient The approximate method based on von Karman's momentum integral equation was extended by Pohlhausen to cover the case of two-dimensional laminar flow with pressure gradient. We choose a coordinate system in which x denotes the distance measured along the surface

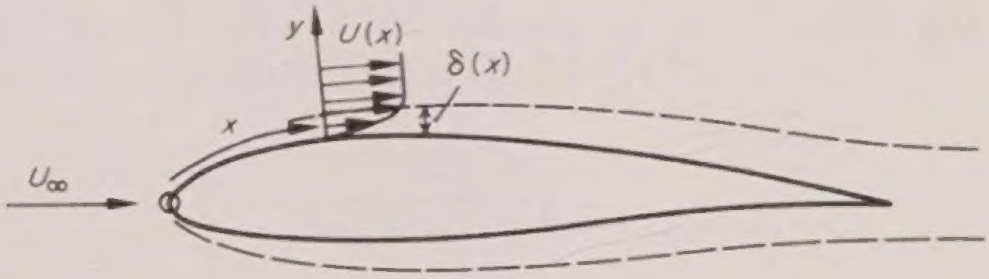


Fig. 4.20

(see Fig. 4.20) and y denotes the normal distance from the surface. The momentum integral equation (4.103) is written in the form

$$U\frac{d\theta}{dx}+(2\theta+\delta^*)U\frac{dU}{dx}=\frac{\tau_0}{\rho}. \tag{4.113}$$

In writing down an expression for the velocity distribution, it can no longer be assumed that the velocity profiles at varying distances downstream of the leading edge are similar. In particular, since $dp/dx \neq 0$, we must allow for the

existence of velocity profiles with and without a point of inflection. Therefore, we assume that the velocity distribution may be expressed as a fourth-degree polynomial equation in $\eta\ (= y/\delta)$, and we write

$$\frac{u}{U} = A_0 + A_1\eta + A_2\eta^2 + A_3\eta^3 + A_4\eta^4 \tag{4.114}$$

where the coefficients may vary with x. To evaluate the five coefficients, we stipulate the five boundary conditions

$$y = 0: \quad u = 0, \quad \frac{\partial^2 u}{\partial y^2} = \frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} = -\frac{U}{\nu}\frac{\mathrm{d}U}{\mathrm{d}x}$$

$$y = \delta: \quad u = U, \quad \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = 0.$$

The first boundary condition at the surface obviously requires that $A_0 = 0$. The second boundary condition at the surface ensures that the velocity profile exhibits a point of inflection when $\mathrm{d}p/\mathrm{d}x > 0$ but has no inflection when $\mathrm{d}p/\mathrm{d}x \leqslant 0$.

We now define the dimensionless parameter

$$\Lambda = \frac{\delta^2}{\nu}\frac{\mathrm{d}U}{\mathrm{d}x} \tag{4.115}$$

and the four remaining coefficients in Eq. (4.114) are then found to be

$$A_1 = 2 + \frac{\Lambda}{6}, \quad A_2 = -\frac{\Lambda}{2}$$

$$A_3 = -2 + \frac{\Lambda}{2}, \quad A_4 = 1 - \frac{\Lambda}{6}.$$

Therefore, the velocity distribution may be written as

$$\frac{u}{U} = (2\eta - 2\eta^3 + \eta^4) + \frac{\Lambda}{6}(\eta - 3\eta^2 + 3\eta^3 - \eta^4)$$

$$= F(\eta) + \Lambda G(\eta) \tag{4.116}$$

where

$$F(\eta) = 2\eta - 2\eta^3 + \eta^4 = 1 - (1+\eta)(1-\eta)^3$$

and

$$G(\eta) = \tfrac{1}{6}(\eta - 3\eta^2 + 3\eta^3 - \eta^4) = \frac{\eta}{6}(1-\eta)^3. \tag{4.117}$$

The velocity distribution expressed by Eq. (4.116) is seen to be a combination of two distributions. The first component, $F(\eta)$, is the polynomial distribution derived in Example (4.4) for zero pressure gradient. The second component, $G(\eta)$, represents the distortion of the velocity profile associated with the pressure gradient, the degree of distortion depending on the magnitude of Λ. Thus, Λ may be regarded as a shape factor. The resulting velocity distributions for various values of Λ are plotted in Fig. 4.21. It follows from the

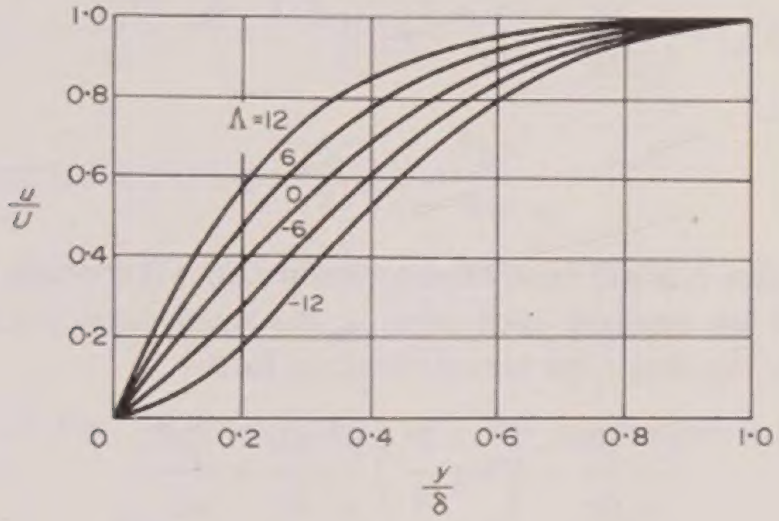


Fig. 4.21 Effect of pressure gradient on velocity distribution

definition of Λ (Eq. (4.115)) and the fact that $\mathrm{d}p/\mathrm{d}x = -\rho U(\mathrm{d}U/\mathrm{d}x)$ that the pressure gradient and the shape factor are related by

$$\Lambda = -\frac{\delta^2}{\mu U}\frac{\mathrm{d}p}{\mathrm{d}x}. \tag{4.118}$$

The calculated velocity distributions indicate that the condition $(\partial u/\partial y)_0 = 0$ (i.e., separation) is achieved when $\Lambda = -12$. For $\Lambda > +12$, the calculation predicts values of $u/U > 1$, which have no physical significance in steady flow. Consequently, we restrict our attention to the range $-12 \leqslant \Lambda \leqslant +12$.

We can now obtain δ^*, θ, and τ_0 in terms of δ and Λ only. Thus,

$$\frac{\delta^*}{\delta} = \int_0^1 \left(1 - \frac{u}{U}\right) \mathrm{d}\eta = \int_0^1 [1 - F(\eta) - \Lambda G(\eta)]\,\mathrm{d}\eta$$

$$= \frac{3}{10} - \frac{\Lambda}{120} \tag{4.119}$$

$$\frac{\theta}{\delta}=\int_0^1 [F(\eta)+\Lambda G(\eta)][1-F(\eta)-\Lambda G(\eta)]\,\mathrm{d}\eta$$

$$=\frac{37}{315}-\frac{\Lambda}{945}-\frac{\Lambda^2}{9072} \tag{4.120}$$

$$\tau_0=\mu\left(\frac{\partial u}{\partial y}\right)_0=\frac{\mu U}{\delta}\left(\frac{\partial(u/U)}{\partial \eta}\right)_0$$

and therefore

$$\frac{\tau_0\delta}{\mu U}=2+\frac{\Lambda}{6}\cdot \tag{4.121}$$

The shape factor Λ is still an unknown quantity since it contains the unknown thickness δ. If we multiply each term in the momentum integral equation (4.113) by $\theta/\nu U$, we obtain the non-dimensional form

$$\frac{U\theta}{\nu}\frac{\mathrm{d}\theta}{\mathrm{d}x}+\left(2+\frac{\delta^*}{\theta}\right)\frac{\theta^2}{\nu}\frac{\mathrm{d}U}{\mathrm{d}x}=\frac{\tau_0\theta}{\mu U}\cdot \tag{4.122}$$

We now define

$$K=\frac{\theta^2}{\nu}\frac{\mathrm{d}U}{\mathrm{d}x} \tag{4.123}$$

and

$$Z=\frac{\theta^2}{\nu} \tag{4.124}$$

so that

$$K=Z\frac{\mathrm{d}U}{\mathrm{d}x}\cdot \tag{4.125}$$

These substitutions and the analysis which follows represent an improvement on the original Pohlhausen method and is due to Holstein and Bohlen. It is seen from Eq. (4.123) that K is a second shape factor which is related to θ in precisely the same way that Λ is related to δ.

From the definitions of Λ and K, it appears that

$$K=\left(\frac{\theta}{\delta}\right)^2\Lambda=\left(\frac{37}{315}-\frac{\Lambda}{945}-\frac{\Lambda^2}{9072}\right)^2\Lambda. \tag{4.126}$$

Writing

$$\frac{\delta^*}{\theta}=\frac{\delta^*/\delta}{\theta/\delta}=\frac{\frac{3}{10}-\frac{\Lambda}{120}}{\frac{37}{315}-\frac{\Lambda}{945}-\frac{\Lambda^2}{9072}}=f_1(K) \tag{4.127}$$

$$\frac{\tau_0\theta}{\mu U}=\frac{\tau_0\delta}{\mu U}\frac{\theta}{\delta}=\left(2+\frac{\Lambda}{6}\right)\left(\frac{37}{315}-\frac{\Lambda}{945}-\frac{\Lambda^2}{9072}\right)=f_2(K) \tag{4.128}$$

and differentiating Eq. (4.124) to give

$$\frac{\theta}{\nu}\frac{\mathrm{d}\theta}{\mathrm{d}x}=\frac{1}{2}\frac{\mathrm{d}Z}{\mathrm{d}x} \tag{4.129}$$

substitution of Eqs. (4.124)-(4.129) into the momentum integral equation (4.122) yields

$$\frac{U}{2}\frac{\mathrm{d}Z}{\mathrm{d}x}+[2+f_1(K)]K=f_2(K)$$

or

$$U\frac{\mathrm{d}Z}{\mathrm{d}x}=2f_2(K)-2K[2+f_1(K)]=F(K). \tag{4.130}$$

Therefore, the momentum integral equation can be written in the abbreviated form

$$\frac{\mathrm{d}Z}{\mathrm{d}x}=\frac{F(K)}{U} \tag{4.131}$$

where K is given by Eq. (4.125).

Equation (4.131) is a first-order non-linear differential equation in Z. Since K, $f_1(K)$, $f_2(K)$, and $F(K)$ can all be written in terms of Λ only, they may be calculated for various values of Λ. Some of the calculated results of Holstein and Bohlen are summarized in Table 4.2.

Table 4.2 Functions for calculation of the laminar boundary layer with pressure gradient (Holstein and Bohlen)

Λ	K	$F(K)$	$f_1(K)$	$f_2(K)$
12	0·0948	− 0·0948	2·250	0·356
11	0·0941	− 0·0912	2·253	0·355
10	0·0919	− 0·0800	2·260	0·351
9	0·0882	− 0·0608	2·273	0·347
8	0·0831	− 0·0335	2·289	0·340
7·052	0·0770	0	2·308	0·332
7	0·0767	0·0021	2·309	0·331
6	0·0689	0·0459	2·333	0·321
5	0·0599	0·0979	2·361	0·310
4	0·0497	0·1579	2·392	0·297
3	0·0385	0·2300	2·427	0·283
2	0·0264	0·3004	2·466	0·268
1	0·0135	0·3820	2·508	0·252
0	0	0·4698	2·554	0·235
− 1	− 0·0140	0·5633	2·604	0·217
− 2	− 0·0284	0·6609	2·647	0·199
− 3	− 0·0429	0·7640	2·716	0·179
− 4	− 0·0575	0·8698	2·779	0·160
− 5	− 0·0720	0·9780	2·847	0·140
− 6	− 0·0862	1·0877	2·921	0·120
− 7	− 0·0999	1·1981	2·999	0·100
− 8	− 0·1130	1·3080	3·085	0·079
− 9	− 0·1254	1·4167	3·176	0·059
− 10	− 0·1369	1·5229	3·276	0·039
− 11	− 0·1474	1·6257	3·383	0·019
− 12	− 0·1567	1·7241	3·500	0

4.7.4 Boundary layer calculations The calculation of the boundary layer parameters can be conveniently undertaken as a step-by-step analysis using the finite-difference form of Eq. (4.131)

$$\Delta Z_{i,j} = \frac{F(K_i)}{U_i} \Delta x_{i,j} \tag{4.132}$$

this giving the increment of Z for the j-th step of the analysis, based on the calculated values of K and U at the end of the preceding step (Fig. 4.22). The calculation begins at the stagnation point and proceeds along the surface until a

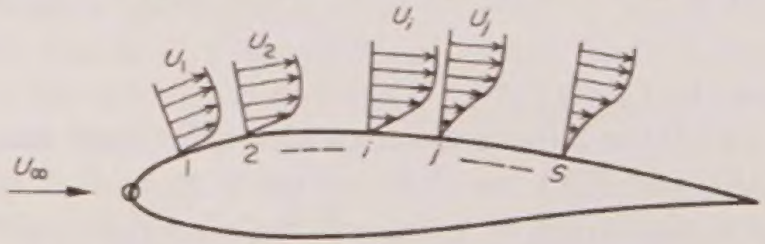


Fig. 4.22

point of separation is reached. The velocity distribution, $U(x)$, outside the boundary layer is assumed to have been obtained by a potential flow analysis.

To commence the calculation, we need to know the initial values of Z and dZ/dx at the stagnation point. From Eq. (4.125), we see that

$$Z_0 = \frac{K_0}{(dU/dx)_0} = \frac{0{\cdot}0770}{(dU/dx)_0} \tag{4.133}$$

but if we attempt to use Eq. (4.132) to determine $(dZ/dx)_0$, we obtain an indeterminate result since both $F(K_0)$ and U_0 are zero. However, if we take the limiting value

$$\left(\frac{dZ}{dx}\right)_0 = \lim_{x \to 0} \left(\frac{\delta Z}{\delta x}\right)$$

it is found that

$$\left(\frac{dZ}{dx}\right)_0 = -0{\cdot}0652 \frac{(d^2 U/dx^2)_0}{(dU/dx)_0}. \tag{4.134}$$

Having decided on the size of step (Δx) to be used, the calculation may proceed with the following routine:

1. Calculate Z_0 and $(dZ/dx)_0$ using Eqs. (4.133) and (4.134), respectively.
2. Calculate

$$Z_1 = Z_0 + \left(\frac{dZ}{dx}\right)_0 \Delta x_{0,1}$$

$$K_1 = Z_1 \left(\frac{dU}{dx}\right)_1$$

3. (i) Obtain Λ from Table 4.2.
   (ii) Calculate θ from Eq. (4.123).
   (iii) Obtain δ^* from Table 4.2 $(f_1(K) = \delta^*/\theta)$.
   (iv) Obtain τ_0 from Table 4.2 $(f_2(K) = \tau_0 \theta/\mu U)$.
   (v) The velocity distribution can be calculated from Eq. (4.116) if required.
4. The second step commences using Eq. (4.132),

$$\Delta Z_{1,2} = \frac{F(K_1)}{U_1} \Delta x_{1,2}.$$

Then

$$Z_2 = Z_1 + \Delta Z_{1,2}$$

and

$$K_2 = Z_2 \left(\frac{dU}{dx}\right)_2.$$

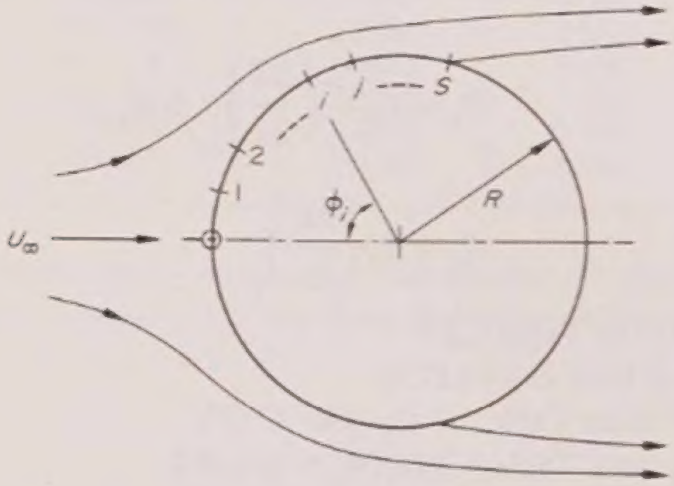
5. The values of Λ, θ, δ^*, τ_0, and the velocity distribution are obtained as before.
6. The third step commences with the use of Eq. (4.132),

$$\Delta Z_{2,3} = \frac{F(K_2)}{U_2} \Delta x_{2,3}$$

. . . etc.

The accuracy of the calculation naturally depends upon the number of steps chosen; this choice may well be dependent on the availability of computer facilities. Adequate calculations can, however, be performed with the aid of a desk calculating machine. The method leads to accurate predictions in regions of negative pressure gradient. With $dp/dx > 0$, accuracy deteriorates as the magnitude of the pressure gradient increases.

Example 4.5 Using a step-by-step calculation, determine the distribution of θ, δ^*, and τ_0 on the surface of a circular cylinder and plot the boundary layer velocity distributions.



The potential flow solution for this case was considered in §3.3.3, which gave

$$U = 2U_\infty \sin\phi.$$

Then

$$\frac{dU}{dx} = \frac{1}{R}\frac{dU}{d\phi} = \frac{2}{R} U_\infty \cos\phi$$

and

$$\frac{d^2U}{dx^2} = \frac{1}{R^2}\frac{d^2U}{d\phi^2} = -\frac{2}{R^2} U_\infty \sin\phi.$$

Following the routine suggested previously and selecting 5° increments in ϕ:

1. $$Z_0 = \frac{0{\cdot}0770}{(dU/dx)_0} = 0{\cdot}0770\,\frac{R}{2U_\infty}$$

$$\left(\frac{dZ}{dx}\right)_0 = -0{\cdot}0652\,\frac{(d^2U/dx^2)_0}{(dU/dx)_0} = 0.$$

2. $$Z_1 = Z_0 + \left(\frac{dZ}{dx}\right)_0 R\,\Delta\phi_{0,1}$$

$$= Z_0 = 0{\cdot}0770\,\frac{R}{2U_\infty}$$

$$K_1 = Z_1\left(\frac{dU}{dx}\right)_1 = 0{\cdot}0770\cos\phi_1$$

$$= 0{\cdot}0770\cos 5^\circ = 0{\cdot}0767.$$

3. (i) $$\Lambda = 7{\cdot}0$$ *

(ii) $$\theta = \sqrt{\frac{\nu K}{(dU/dx)}} = \sqrt{\frac{0{\cdot}0767\nu R}{2U_\infty\cos\phi}}$$

$$\therefore\quad \frac{\theta_1}{R}\sqrt{\frac{U_\infty R}{\nu}} = \sqrt{\frac{0{\cdot}0767}{2\cos 5^\circ}} = 0{\cdot}1960$$ **

(iii) $$f_1(K) = \frac{\delta^*}{\theta} = 2{\cdot}309$$

$$\therefore\quad \frac{\delta^*}{R}\sqrt{\frac{U_\infty R}{\nu}} = 2{\cdot}309 \times 0{\cdot}1960 = 0{\cdot}4526$$ ***

(iv) $$f_2(K) = \frac{\tau_0\,\theta}{\mu U} = 0{\cdot}331$$

$$\therefore\quad \tau_0 = 0{\cdot}331\,\frac{\mu U}{\theta} = \frac{0{\cdot}331}{0{\cdot}196}\,\frac{\mu U}{R}\sqrt{\frac{U_\infty R}{\nu}}$$

$$\therefore\quad \frac{\tau_0}{\frac{1}{2}\rho U_\infty^2} = \frac{2\times 0{\cdot}331}{0{\cdot}196}\,\frac{U}{U_\infty}\sqrt{\frac{\nu}{U_\infty R}}$$

$$\therefore\quad \frac{\tau_0}{\frac{1}{2}\rho U_\infty^2}\sqrt{\frac{U_\infty R}{\nu}} = \frac{2\times 0{\cdot}331}{0{\cdot}196}\,2\cos 5^\circ$$

$$= 0{\cdot}589.$$ ****

(v) The velocity distribution is calculated from Eq. (4.116).

4. For the second step:

$$\Delta Z_{1,2} = \frac{F(K_1)}{U_1} R \Delta\phi_{1,2}$$

$$= \frac{0{\cdot}0021\, R}{2U_\infty \sin 5^\circ} \frac{5}{57{\cdot}3} = 0{\cdot}0021 \frac{R}{2U_\infty}$$

$$\therefore \quad Z_2 = Z_1 + \Delta Z_{1,2} = 0{\cdot}0791 \frac{R}{2U_\infty}$$

$$\therefore \quad K_2 = Z_2 \left(\frac{dU}{dx}\right)_2 = 0{\cdot}0791 \cos 10^\circ$$

$$= 0{\cdot}0767.$$

5. The values of $\Lambda, \theta, \delta^*$, and τ_0 are then obtained as for the first step.
6. The procedure is continued until a point is reached for which $\tau_0 = 0$ (i.e., separation). The results of the complete calculation are plotted in Fig. 4.23. The results of the exact solution are also included for comparison.

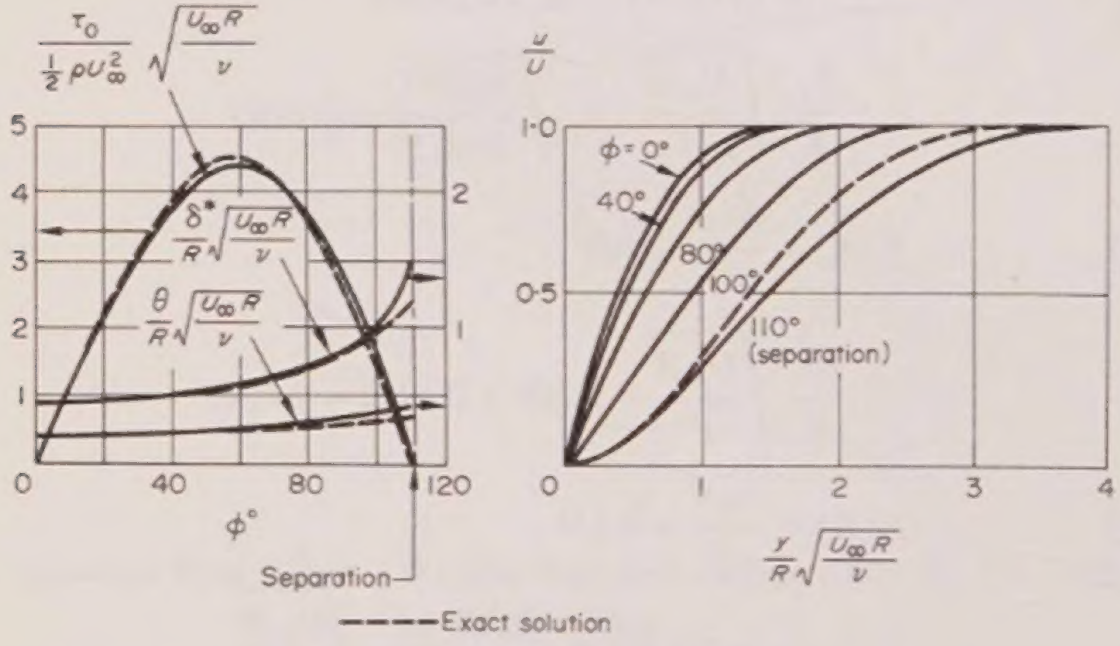


Fig. 4.23 Approximate calculation of the laminar boundary layer on a circular cylinder